

Absolutely continuous spectrum implies ballistic transport for quantum particles in a random potential on tree graphs

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We discuss the dynamical implications of the recent proof that for a quantum particle in a random potential on a regular tree graph absolutely continuous (*ac*) spectrum occurs non-perturbatively through rare fluctuation-enabled resonances. The main result is spelled in the title.

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I. INTRODUCTION

Progress was recently made in the understanding of the spectra of Schrödinger operators with random potential on tree graphs. In particular, it was found that absolutely continuous (*ac*) spectrum is quite robust there, and already at weak disorder *ac* spectrum appears in regimes of extremely low density of states.^{2,3} The proof suggests that in such regimes the spread of the wave function occurs by tunneling which is enabled by rare resonances. It is natural to ask how do wave packets with energies limited to such a regime spread, and at what rate does the distribution of the particle's distance from its starting point grow. Our purpose here is to answer the latter question.

A. Bounds on quantum dynamics

The quantum dynamics of a particle moving on a graph, which is composed of a vertex set \mathcal{G} and an edge set \mathcal{E} , is generated by a Schrödinger operator of the form

$$(H\psi)(x) = - \sum_{(x,y) \in \mathcal{E}} \psi(y) + V(x)\psi(x). \quad (\text{I.1})$$

If the vertex degree of the graph is uniformly bounded (in which case the first term in (I.1) defines a bounded operator) then for any real valued potential V the operator H is self-adjoint operator on an appropriate domain in the Hilbert space $\ell^2(\mathcal{G})$. Under the unitary time-evolution generated by H the probability of the particle having the position $x \in \mathcal{G}$ at time $t > 0$ after it was started at a quantum state ψ is given by:

$$P_{\psi,t}(x) := |(e^{-itH}\psi)(x)|^2. \quad (\text{I.2})$$

While the probabilistic interpretation of $P_{\psi,t}(x)$ is limited to vectors with norm $\|\psi\| = 1$, it is convenient for us to extend the above symbol to all ψ regardless of their normalization.

Instead of investigating $P_{\psi,t}(x)$ directly, it is often easier to study the time-averaged transition probability

$$\widehat{P}_{\psi,\eta}(x) := 2\eta \int_0^\infty e^{-2\eta t} P_{\psi,t}(x) dt = \frac{\eta}{\pi} \int |((H - E - i\eta)^{-1}\psi)(x)|^2 dE, \quad (\text{I.3})$$

with inverse time-parameter $\eta > 0$. The equality, which is based on the spectral theorem and Plancherel's identity, links the long-time averages ($\eta \downarrow 0$) of the probability distribution $\widehat{P}_{\psi,\eta}(\cdot)$ with properties of the operator's Green function $G(x, y; \zeta)$ on which our analysis will focus.

The rate of growth of the distance travelled is conveniently described by the moments of the corresponding distributions, which are defined as

$$M_\psi(\beta, t) := \sum_{x \in \mathcal{G}} |x|^\beta P_{\psi,t}(x) \quad \text{and} \quad \widehat{M}_\psi(\beta, \eta) := \sum_{x \in \mathcal{G}} |x|^\beta \widehat{P}_{\psi,\eta}(x). \quad (\text{I.4})$$

Here and in the following $|x| := d(x, 0)$ denotes the graph-distance of the vertex x to some fixed, but arbitrary vertex $0 \in \mathcal{G}$.

To place our results in their natural context, let us recall some points of reference on the relation of quantum dynamics with the spectral properties of their generator.

1. In the presence of disorder, in particular for random potentials, there may exist subspaces of $\ell^2(\mathcal{G})$ over which one finds different behavior. For functions ψ which are spanned by localized states, the moments $M_\psi(\beta, t)$ remain bounded uniformly in time.¹ For functions ψ in the subspace generated by extended (generalized) eigenfunctions, the moments increase at least in the average sense that $\widehat{M}_\psi(\beta, \eta) \rightarrow \infty$ for $\eta \rightarrow 0$ and any $\beta > 0$ (by the RAGE theorem; cf. Ref. 11).
2. In the physical picture of delocalization in the presence of disorder, which was advanced by D. Thouless and collaborators, it is generally expected (though the statement still remains unproven) that in the corresponding spectral regime the probability distribution $P_\psi(x, t)$ spreads at a rate corresponding to diffusion (cf. Ref. 13). In the finite dimensional case, of $\mathcal{G} = \mathbb{Z}^d$, that translates to:

$$M_\psi(\beta, t) \sim t^{r\beta}, \quad \text{and} \quad \widehat{M}_\psi(\beta, \eta) \sim \eta^{-r\beta}, \quad \text{with } r = 1/2. \quad (\text{I.5})$$

This is in contrast to the ballistic motion for which $r = 1$, as is the case in the absence of disorder, with V either constant or periodic in case of \mathbb{Z}^d , or radially periodic on regular tree graphs. It is however relevant here to note that in the hyperbolic geometry of a regular tree the classical diffusion also spreads ballistically, since at each instance there are more directions at which $|x|$ would increase than the one direction at which it goes down.

3. A general upper bound can be obtained from the observation that on any graph $(\mathcal{G}, \mathcal{E})$ with a uniformly bounded vertex degree, the distance travelled does not increase faster than at some finite speed $\hat{v} < \infty$, in the sense that the probability for faster growth decays exponentially:

$$\Pr_{\delta_u, t}(d(x, u) > vt) := \sum_{x: d(x, u) > vt} P_{\delta_u, t}(x) \leq e^{-\mu t(v - \hat{v})} \quad (\text{I.6})$$

at some $\mu > 0$, where the initial state is taken to be the normalized function localized at (an arbitrary) vertex $u \in \mathcal{G}$, and $d(x, u)$ is the distance between the two sites $x, u \in \mathcal{G}$. This bound holds regardless of the potential V , and in particular it implies the ballistic upper bounds:

$$\widehat{M}_\psi(\beta, \eta) \leq C(\beta, \psi) \eta^{-\beta} \quad (\text{I.7})$$

for all normalized $\psi \in \ell^2(\mathcal{G})$ with $\sum_{x \in \mathcal{G}} |x|^\beta |\psi(x)|^2 < \infty$. For completeness, a proof of (I.6) is included in the appendix, where we also comment on its relation with the Lieb-Robinson bounds.²⁰

4. Lower bounds on the moments can be obtained by estimating the probability of lingering:

$$\Pr_{\psi, t}(|x| < b t^r) := \sum_{|x| < b t^r} P_\psi(x, t) \quad \text{and} \quad \widehat{\Pr}_{\psi, \eta}(|x| < b \eta^{-r}) := \sum_{|x| < b \eta^{-r}} \widehat{P}_{\psi, \eta}(x, t). \quad (\text{I.8})$$

The afore-mentioned RAGE theorem implies that for any state ψ within which H has only continuous spectrum:

$$\lim_{\eta \rightarrow 0} \widehat{\Pr}_{\psi, \eta}(|x| < b) = 0 \quad (\text{I.9})$$

for any finite b (and $r = 0$). Stronger general bounds, due originally to I. Guarneri (with generalization found in Refs. 10, 15, and 19), are based on the finer distinction among spectral types, classified by the Hausdorff dimension of the spectrum and more precisely by the degree of Hölder continuity of the spectral measure associated with ψ . That measure is said to be uniformly α -Hölder continuous, for $\alpha \in (0, 1]$, if for all Borel sets $I \subset \mathbb{R}$ of Lebesgue measure $|I| \leq 1$ one has $\mu_\psi(I) \leq C_\psi |I|^\alpha$, at a common value of C_ψ . For $\mathcal{G} = \mathbb{Z}^d$, Guarneri¹² proved that in such cases $\widehat{P}_{\psi, \eta}(x) \leq C \eta^\alpha$ and thus

$$\widehat{\Pr}_{\psi, \eta}(|x| < b \eta^{-\alpha/d}) \leq C_d b^d. \quad (\text{I.10})$$

This directly implies that for any $\beta > 0$ (I.5) can hold only with $r \geq \alpha/d$, since selecting b so that $C_d b^d = 1/2$ one gets:

$$\widehat{M}_\psi(\beta, \eta) \geq \frac{b^\beta}{2} \eta^{-\beta\alpha/d}. \quad (\text{I.11})$$

There are operators with ac spectrum, corresponding to $\alpha = 1$, for which the Guarneri bound is almost saturated.⁷ However, this lower bound diminishes with dimension and it provides no information for operators on tree graphs (which correspond to $d = \infty$).

In this note we focus on the case $(\mathcal{G}, \mathcal{E})$ is a regular rooted tree graph and the operator (I.1) is random and known to have a regime of ac spectrum. Our main result is that the moments grow ballistically, that is $\widehat{M}_\psi(\beta, \eta)$ obey not only an upper bound but also a lower bound with $r = 1$.

B. Statement of the main result

The main topic of this note are operators of the form (I.1) on the Hilbert space over a regular rooted tree graph, whose vertex set we denote \mathcal{T} , in which every vertex aside from the root 0 has $K + 1$ neighbors with $K \geq 2$. We take the potential $V : \mathcal{T} \rightarrow \mathbb{R}$ to be random, with a distribution described by:

- A1. $V(x)$, $x \in \mathcal{T}$, are independent, identically distributed random variables,
- A2. the probability distribution of the potential at a site is ac, $\mathbb{P}(V(x) \in dv) = \varrho(v) dv$, with a bounded probability density $\varrho \in L^\infty(\mathbb{R})$ satisfying:
 - (a) the moment condition: $\mathbb{E}[|V(0)|^6] := \int_{\mathbb{R}} v^6 \varrho(v) dv < \infty$;
 - (b) the local upper bounds:

$$\sup_{|v-\zeta|<1} \varrho(v) \leq c \int_{|v-\zeta|<1} \varrho(v) dv \quad (\text{I.12})$$

for all $\zeta \in \mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im } z > 0\}$. [The supremum over the empty set is set by convention to minus infinity.]

Let us recall some known facts for random Schrödinger operators H of the form (I.1).^{9,14,22} By ergodicity arguments the spectrum $\Sigma(H)$ is almost surely given by a non-random closed set. Properly formulated, that also holds for the spectra corresponding to the different spectral components in the Lebesgue decomposition of the spectral measure, i.e., the absolutely continuous (*ac*), singular continuous (*sc*), and pure point (*pp*) spectrum for which we shall use the capital letter $\Sigma_\#(H)$, with $\#$ standing for *ac*, *sc*, or *pp*.

Instead of the closed set $\Sigma_{ac}(H)$, we will rather focus on a measure-theoretic support of the *ac* density of the spectral measure. To describe the latter, one may start from two generally valid facts: *i.* for Lebesgue almost every $E \in \mathbb{R}$ the limit $G(0, 0; E + i0) = \lim_{\eta \downarrow 0} G(0, 0; E + i\eta)$ exists almost surely, and *ii.* the *ac* component of the spectral measure associated with the vector δ_0 is $\text{Im } G(0, 0; E + i0) dE/\pi$. We then define:

$$\sigma_{ac}(H) := \{E \in \mathbb{R} : \mathbb{P}(\text{Im } G(0, 0; E + i0) \neq 0) > 0\}. \quad (\text{I.13})$$

For reasons which are explained in Appendix A, this notion of the *ac* spectrum is better suited for our purpose than $\Sigma_{ac}(H)$. Both sets are non-random, and possibly differ in only a set of zero Lebesgue measure. However, that was not established yet. It may be added that at least on tree graphs, for each energy E the probability in (I.13) is either zero or one. It follows that the non-random set $\sigma_{ac}(H)$ is also the support of the *ac* component of the spectrum for almost every realization of the randomness.

While it is rather straightforward to prove that $\Sigma(H) = [-2\sqrt{K}, 2\sqrt{K}] + \text{supp } \varrho$, determining the spectral components is usually harder. In the tree situation, the different spectral regimes can be characterized (almost completely) by a function $\varphi(E; 1)$, in terms of which:

1. if $\varphi(E; 1) < \log K$ for all energies E in some interval $I \subset \Sigma(H)$ then $I \subset \Sigma_{pp}(H)$.
2. if $\varphi(E; 1) > \log K$ for almost all energies E in a measurable subset $S \subset \Sigma(H)$ then $S \subset \sigma_{ac}(H)$ up to a difference of zero Lebesgue measure, and furthermore H has only *ac* spectrum in S .

The function is constructed as the boundary value $\varphi(E; 1) := \lim_{s \uparrow 1} \varphi(E; s)$ of a large deviation free energy function defined in (II.14) below.⁴ We could not calculate $\varphi(E; 1)$ explicitly, or prove regularity and continuity, except for some partial statements. However, its analysis and that of the Lyapunov exponent, which bounds $\varphi(E; 1)$ from below, yields the following picture:

1. In case of unbounded potentials pure point spectrum emerges at extreme energies ($|E| > K + 1$) for small disorder¹ and it covers the whole spectrum for large disorder, as was previously indicated in the work of R. Abou-Chacra, P. Anderson, and D. Thouless.⁵
2. The above criterion was recently used to show that *ac* spectrum emerges for arbitrarily small disorder well beyond the spectrum of the adjacency operator, in the regime $[-(K + 1), (K + 1)] \cap \Sigma(H)$ (Refs. 2 and 4). Earlier, the persistence of *ac* spectrum within the spectrum of the adjacency operator, i.e. $[-2\sqrt{K}, 2\sqrt{K}] \cap \Sigma_{ac}(H) \neq \emptyset$, was established by A. Klein¹⁶, through a continuity argument (which was recently extended to the Bethe strip¹⁸).
3. Somewhat surprisingly, for bounded random potentials the criterion allowed also to establish that at weak disorder at its edges the spectrum is purely *ac*. In particular, in that case there is no localization at band edges.³

The non-perturbative emergence of ac spectrum well beyond the spectrum of the adjacency operator has been explained in terms of a resonance mechanism for which the exponential growth of the volume in a tree graph plays an important role (cf. Refs. 4 and 2 for a short summary). Along with that goes a picture of extended states which are localized at an infinite collection of ‘resonating vertices’.

In this context, it natural to ask about the dynamical behavior of the states within the ac spectrum. It was already shown by A. Klein¹⁷ that in the regime where the persistence of ac spectrum at weak disorder could be established by a continuity argument, the averaged dynamics is ballistic. We now prove that ballistic behavior extends to the full regime of ac spectrum, including the region where the dynamics appear to be dominated by tunneling events. Following is the key bound.

Theorem I.1. *Let H be an operator of the form (I.1) on a regular rooted tree graph with a random potential satisfying the above conditions A1–2. Then for any initial state of the form $\psi = f(H)\delta_0$, with a measurable function $f \in L^2(\mathbb{R})$ with $\{f \neq 0\} \subset \sigma_{ac}(H)$, and all $b > 0$:*

$$\mathbb{E} \left[\widehat{\Pr}_{\psi, \eta}(|x| < b\eta^{-1}) \right] \leq C(f)b + o(\eta). \quad (\text{I.14})$$

with some $C(f) < \infty$, and $o(\eta)$ a quantity which vanishes for $\eta \rightarrow 0$.

At the risk of partial repetition, we close this section with several remarks:

1. The complementary bounds (I.14) and (I.6) show that throughout the ac spectrum the (doubly averaged) quantum time evolution in a random potential on a tree is ballistic.
2. The above notwithstanding, Theorem I.1 is also consistent with the afore mentioned conjecture of diffusive evolution, since on regular tree graphs the classical diffusion spreads ballistically.
3. In extending the previous proof of ballistic behavior from the perturbative regime (of small randomness and energies within the spectrum of the adjacency operator)¹⁷ to the full region of ac states, Theorem I.1 excludes the possibility of another dynamical behavior in the regime where ac spectrum is caused by rare resonances. This includes energy regimes where the density of states is extremely low, with Lifshitz tail asymptotics (as discussed in Ref. 2). The diffusion constant in this regime, for which the proof yields a lower bound involving $\mathbb{E}(\|f(H)\delta_0\|^2)$, should be correspondingly small.
4. One may ask whether there are operators similar to (I.1) on tree graphs, for which wave packets of states within the continuous spectrum spread at a slower than ballistic rate. Sub-ballistic rates are known to occur in classical random walks on trees in certain ‘random conductance models’. In these random walk models the Laplacian is replaced by an operator with random (though still nearest-neighbor) hopping amplitudes, whose distribution extends down to zero (cf. Ref. 8).

II. PROOF OF BALLISTIC TRANSPORT ON TREE GRAPHS

Aside from some functional analytic manipulations, the main new ingredient used here in the proof of Theorem I.1 is the statement that throughout any bounded measurable subset of $\sigma_{ac}(H)$ the Green function’s second moments obeys, for all $x \in \mathcal{T}$:

$$\operatorname{ess\,sup}_{\zeta \in I+i(0,1]} \mathbb{E} [|G(0, x; \zeta)|^2] \leq \frac{C_+(I)}{K|x|}, \quad (\text{II.1})$$

with $C_+(I) < \infty$. (Here and in the following $\operatorname{ess\,sup}$ stands for the Lebesgue-essential supremum.)

In Lemma II.2 we prove that (II.1) is implied by the statement that for any bounded subset $I \subset \sigma_{ac}(H)$:

$$\operatorname{ess\,sup}_{\zeta \in I+i(0,1]} \mathbb{E} [(\operatorname{Im} G(0, 0; \zeta))^{-1-\delta}] < \infty. \quad (\text{II.2})$$

at some $\delta > 0$ (which does not depend of I). The derivation of (II.2) is the subject of Theorem II.4.

Assuming the validity of (II.1) the proof of Theorem I.1 is rather elementary, while the proofs of Lemma II.2 and Theorem II.4 are increasingly more involved. We shall therefore establish the above statements in this order: first show how (II.1) implies the bound claimed in Theorem I.1, then through Lemma II.2 show how (II.2) implies (II.1), and finally (in part C) establish Theorem II.4 and through it (II.2).

A. A conditional proof of the main result

We start by relating $\mathbb{E}[\widehat{P}_{f(H)\varphi,\eta}]$ with an even more convenient quantity. The difference between the two is the vanishing term, $o(\eta)$, in (I.14). As a technical tool, we will employ the Wegner estimate which guarantees the absolute continuity of the average of the spectral measure μ_φ of the random operator associated with any (a priori fixed) vector $\varphi \in \ell^2(\mathcal{G})$:

$$\mathbb{E}[\mu_\varphi(dE)] \leq C_W \|\varphi\|^2 dE, \quad (\text{II.3})$$

where C_W is a finite constant.^{24,25}

Lemma II.1. *For a random operator satisfying the Wegner estimate (II.3), any $f \in L^2(\mathbb{R})$ and any $\varphi \in \ell^2(\mathcal{G})$ let*

$$K_{\varphi,f,\eta}(x) := \frac{\eta}{\pi} \int |f(E)|^2 |((H - E - i\eta)^{-1}\varphi)(x)|^2 dE. \quad (\text{II.4})$$

Then the following ℓ^1 convergence holds

$$\lim_{\eta \downarrow 0} \sum_{x \in \mathcal{G}} \left| \mathbb{E}[\widehat{P}_{f(H)\varphi,\eta}(x; \eta)] - \mathbb{E}[K_{\varphi,f,\eta}(x)] \right| = 0. \quad (\text{II.5})$$

Proof. The Wegner's estimate (II.3) guarantees the uniform boundedness of the ℓ^1 -norm of $K_{\varphi,f,\eta}$. Namely, abbreviating $\delta_\eta(x) := \pi^{-1} \text{Im}(x - i\eta)^{-1}$, which is an approximate δ -function, the spectral representation yields:

$$0 \leq \sum_{x \in \mathcal{G}} K_{\varphi,f,\eta}(x) = \int |f(E)|^2 \int \delta_\eta(E' - E) \mathbb{E}[\mu_\varphi(dE')] dE \leq C_W \|\varphi\|^2 \|f\|_2^2. \quad (\text{II.6})$$

Using (I.3), the triangle inequality and the Cauchy-Schwarz inequality, it is not hard to see that

$$\sum_{x \in \mathcal{G}} \left| \mathbb{E}[\widehat{P}_{f(H)\varphi,\eta}(x)] - \mathbb{E}[K_{\varphi,f,\eta}(x)] \right| \leq Q(\eta) + 2\sqrt{C_W} \|\varphi\| \|f\| \sqrt{Q(\eta)}, \quad (\text{II.7})$$

where

$$\begin{aligned} Q(\eta) &:= \frac{\eta}{\pi} \sum_{x \in \mathcal{G}} \int \mathbb{E} \left[\left| ((H - E - i\eta)^{-1} f(H)\varphi)(x) - f(E) ((H - E - i\eta)^{-1}\varphi)(x) \right|^2 \right] dE \\ &= \mathbb{E} \left[\int \int \delta_\eta(E' - E) |f(E') - f(E)|^2 dE \mu_\varphi(dE') \right]. \end{aligned} \quad (\text{II.8})$$

The equality is again based on the spectral representation. Applying now the Wegner bound (II.3) again, we get:

$$\begin{aligned} Q(\eta) &\leq C_W \|\varphi\|^2 \int \int \delta_\eta(E' - E) |f(E') - f(E)|^2 dE dE' \\ &= 2C_W \|\varphi\|^2 \left[\int |f(E)|^2 dE - \int_{\mathbb{R}} \int \bar{f}(E') \delta_\eta(E' - E) f(E) dE dE' \right]. \end{aligned} \quad (\text{II.9})$$

In the limit $\eta \downarrow 0$ the above quantity vanishes due to the weak convergence to identity of the operator in $L^2(\mathbb{R})$ whose kernel is $\delta_\eta(E' - E)$. The latter statement is easily seen in the Fourier representation, where the operator corresponds to multiplication by $e^{-|\tau|\eta}$, with τ denoting the Fourier transform variable. \square

Assuming now the bound (II.1), which is proven below independently of the next argument, we proceed to the first of the three steps outlined above.

Conditional proof of Theorem I.1. By Lemma II.1 (and using the notation introduced in its proof)

$$\left| \mathbb{E}[\widehat{\text{Pr}}_{\psi,\eta}(|x| < R)] - \sum_{x: |x| < R} \mathbb{E}[K_{\delta_0,f,\eta}(x)] \right| = o(\eta). \quad (\text{II.10})$$

In the special case $\varphi = \delta_0$ one has

$$K_{\delta_0, f, \eta}(x) = \frac{\eta}{\pi} \int |f(E)|^2 |G(x, 0; E + i\eta)|^2 dE. \quad (\text{II.11})$$

The estimate (II.1) with $I_f := \{E \in \mathbb{R} : f(E) \neq 0\}$ yields

$$\eta \sum_{|x| < R} \mathbb{E} [|G(x, 0; E + i\eta)|^2] \leq C_+(I_f) \eta \sum_{n=0}^{R-1} 1 = C_+(I_f) \eta R. \quad (\text{II.12})$$

We thus have

$$\sum_{|x| < R} \mathbb{E} [K_{\delta_0, f, \eta}(x)] \leq \frac{C_+(I_f) \|f\|_2^2}{\pi} \eta R, \quad (\text{II.13})$$

The proof of Theorem I.1 is concluded by combining the bounds (II.10) and (II.13), and choosing $R = b \eta^{-1}$. \square

B. The significance of the negative moments of the Green function

Our next goal is to show that (II.1) follows from (II.2). In the proof we shall make use of some of the structure which was developed in Ref. 4. It was proven there that the following limit, which defines what is called there the free-energy function, exists and is finite for any $s \in [0, \infty)$ and $\zeta \in \mathbb{C}^+$

$$\varphi(s; \zeta) := \lim_{|x| \rightarrow \infty} \frac{1}{|x|} \log \mathbb{E} [|G(0, x; \zeta)|^s]. \quad (\text{II.14})$$

Some useful properties (taken from Section 3 in Ref. 4) are:

1. For any $s \in [0, 2]$ and any $\zeta \in \mathbb{C}^+$:

$$\varphi(s; \zeta) \leq -s \log K. \quad (\text{II.15})$$

In fact, the inequality is strict for any $\zeta \in \mathbb{C}^+$.

2. For any $s \in [0, 1)$ and $x \in \mathcal{T}$ the following ‘finite-volume bounds’ hold

$$C_-(s; \zeta) e^{\varphi(s; \zeta) |x|} \leq \mathbb{E} [|G(0, x; \zeta)|^s] \leq C_+(s; \zeta) e^{\varphi(s; \zeta) |x|} \quad (\text{II.16})$$

with $C_{\pm}(s; \zeta) \in (0, \infty)$, which at fixed $s \in [0, 1)$ are bounded uniformly in $\zeta \in [-E, E] + i(0, 1]$ for any $0 \leq E < \infty$.

In general one does not expect the bounds (II.16) to hold beyond $s = 1$, since, for instance, for energies in the regime of pure point spectrum: $\mathbb{E} [|G(0, x; E + i0)|] = \infty$. However, as we assert next (based on the argument provided in Ref. 4) these bounds do extend to all energies at which the imaginary part of the resolvent has a finite inverse moment of power greater than one. As will be shown in Theorem II.4 below, this includes the entire ac spectrum.

Lemma II.2. *Under the assumptions of Theorem I.1, if for some bounded measurable set $I \subset \mathbb{R}$ and some $\epsilon > 0$*

$$\operatorname{ess\,sup}_{\zeta \in I + i(0, 1]} \mathbb{E} \left[(\operatorname{Im} G(0, 0; \zeta))^{-1-\epsilon} \right] < \infty. \quad (\text{II.17})$$

then for almost all $E \in I$ and all $\eta \in (0, 1]$:

$$C_- e^{\varphi(2; E + i\eta) |x|} \leq \mathbb{E} [|G(0, x; E + i\eta)|^2] \leq C_+ e^{\varphi(2; E + i\eta) |x|} \quad (\text{II.18})$$

with some $C_{\pm} \in (0, \infty)$.

Of main interest for us is the *upper bound* in (II.18), which together with (II.15) yields (II.1).

For the proof of Lemma II.2, which proceeds essentially along the lines of Theorem 3.2 in Ref. 4, we recall some special properties of the Green function in a tree geometry:

1. For any $x \in \mathcal{T} \setminus \{0\}$ we denote by $\mathcal{P}_{0,x}$ the unique path connecting 0 and x . For each vertex $u \in \mathcal{P}_{0,x}$ other than the path's endpoints 0 and x , the vertex set $\mathcal{T}_u := \mathcal{T} \setminus \{u\}$ decomposes into $K+1$ trees rooted at the neighbors $v \in \mathcal{N}_u$ of u . We denote by $G^{\mathcal{T}_u}$ the Green function of the natural restriction of H to $\ell^2(\mathcal{T}_u)$. The diagonal element of the Green function at u can then be written as

$$G(u, u; \zeta) = \left(V(u) - \zeta - \sum_{v \in \mathcal{N}_u} G^{\mathcal{T}_u}(v, v; \zeta) \right)^{-1}, \quad (\text{II.19})$$

and the Green function between 0 and x admits the factorization:

$$G(0, x; \zeta) = G^{\mathcal{T}_u}(0, u_-; \zeta) G(u, u; \zeta) G^{\mathcal{T}_u}(u_+, x; \zeta), \quad (\text{II.20})$$

where u_{\pm} is the forward/backward neighbor of u on the path $\mathcal{P}_{0,x}$.

2. Conditioning on the sigma algebra \mathcal{A}_u generated by the random variables $\{V(y) \mid y \in \mathcal{T}_u\}$, we thus obtain:

$$\begin{aligned} \mathbb{E} \left[|G(u, u; \zeta)|^2 \mid \mathcal{A}_u \right] &\leq \|\varrho\|_{\infty} \mathbb{E} \left[\int \left| v - \zeta - \sum_{v \in \mathcal{N}_u} G^{\mathcal{T}_u}(v, v; \zeta) \right|^{-2} dv \right] \\ &\leq \pi \|\varrho\|_{\infty} \mathbb{E} \left[\left(\text{Im } G^{\mathcal{T}_u}(v, v; \zeta) \right)^{-1} \right] \end{aligned} \quad (\text{II.21})$$

where $v \in \mathcal{N}_u \setminus \{u_{-}\}$. Note that in this case $G^{\mathcal{T}_u}(v, v; \zeta)$ has the same distribution as $G(0, 0; \zeta)$.

Proof of Lemma II.2. As explained in Ref. 4, the claim follows from sub-/supermultiplicative bounds of the form:

$$c_-^{-1} \leq \mathbb{E}_2 \left[|G(u, u; \zeta)|^2 \right] \leq c_+, \quad (\text{II.22})$$

for all $\text{Re } \zeta \in I$ and $\text{Im } \zeta \in (0, 1]$, where we use the abbreviation

$$\mathbb{E}_2 [\cdot] := \frac{\mathbb{E} \left[|G^{\mathcal{T}_u}(0, x_-; \zeta)|^2 |G^{\mathcal{T}_u}(u_+, x_-; \zeta)|^2 (\cdot) \right]}{\mathbb{E} \left[|G^{\mathcal{T}_u}(0, u_-; \zeta)|^2 \right] \mathbb{E} \left[|G^{\mathcal{T}_u}(u_+, x_-; \zeta)|^2 \right]} \quad (\text{II.23})$$

for a weighted, or ‘tilted’ - in the language of large deviations theory, expectation. [Note that the latter depends on various parameters such as ζ and the involved vertices which are suppressed in the notation.]

The claimed bounds (II.22) rely on the factorization property (II.20). Namely, by virtue of (II.19) and (II.21), the upper bound holds with $c_+ = \pi \|\varrho\|_{\infty} \text{ess sup}_{\zeta \in I+i(0,1]} \mathbb{E} \left[\left(\text{Im } G(0, 0; \zeta) \right)^{-1} \right]$.

The proof of the lower bound also rests on (II.20) and (II.19) which yield for any $t > 0$:

$$\mathbb{E}_2 \left[|G(u, u; \zeta)|^2 \right] \geq \int \frac{\varrho(v) dv}{[|v| + |\zeta| + (K+1)t]^2} \prod_{w \in \mathcal{N}_u} \mathbb{P}_2 (|G^{\mathcal{T}_u}(w, w; \zeta)| \leq t), \quad (\text{II.24})$$

where we have used the independence (also under the tilted measure) of the Green functions $G^{\mathcal{T}_u}$ for different $w \in \mathcal{N}_u$. For $w \notin \{u_+, u_{-}\}$, one has

$$\mathbb{P}_2 (|G^{\mathcal{T}_u}(w, w; \zeta)| \leq t) = \mathbb{P} (|G^{\mathcal{T}_u}(w, w; \zeta)| \leq t) \geq 1 - \frac{1}{ts} \sup_{\zeta \in \mathbb{C}^+} \int \frac{\varrho(v) dv}{|v - \zeta|^s} \geq \frac{1}{2}, \quad (\text{II.25})$$

provided $t > 0$ is chosen large enough. In case $w \in \{u_+, u_{-}\}$, we use the representation $G^{\mathcal{T}_u}(w, w; \zeta) = (V(w) - \zeta - \sum_{v \in \mathcal{N}_w \setminus \{u\}} G^{\mathcal{T}_v}(v, v; \zeta))^{-1}$ and choose $v \in \mathcal{N}_w \setminus \{u\}$, which does not coincide with any of the vertices on the paths $\mathcal{P}_{0,x}$. The proof is completed by applying the principle stated in the Lemma which follows. In applying it, we take $G^{\mathcal{T}_v}(v, v; \zeta)$ be the random variable γ , and note that the constant C which Lemma II.3 yields is independent of ζ and the vertex v . Hence there is some $t > 0$ such that $\mathbb{P}_2 (|G^{\mathcal{T}_u}(v, v; \zeta)| \leq t) \geq \frac{1}{2}$ for all $\zeta \in \mathbb{C}^+$. This completes the proof of the lower bound. \square

The lower bound in the previous lemma used the following explicit decoupling-type estimate.

Lemma II.3. Assume that:

1. p is a probability measure on \mathbb{C}^+ such that $(M_{1+\epsilon} := \int_{\mathbb{C}^+} (\text{Im } \gamma)^{-1-\epsilon} p(d\gamma) < \infty$ for some $\epsilon > 0$,
2. ϱ is a probability density on \mathbb{R} which satisfies (I.12) for some $c \in (0, \infty)$ and all $\zeta \in \mathbb{C}^+$.

Then for all $t \geq 1$ and all $\zeta \in \mathbb{C}^+$:

$$\int_{\mathbb{C}^+} \int_{\mathbb{R}} 1_{|v-\zeta-\gamma| \leq \frac{1}{t}} \frac{\varrho(v) dv p(d\gamma)}{|v-\zeta-\gamma|^2} \leq \frac{C}{t^\epsilon} \int_{\mathbb{C}^+} \int_{\mathbb{R}} \frac{\varrho(v) dv p(d\gamma)}{|v-\zeta-\gamma|^2}, \quad (\text{II.26})$$

where $C := c \pi M_{1+\epsilon} \inf_{T>0} \frac{(1+T)^2}{p(|\gamma| \leq T)}$.

Proof. We start by estimating the left side by

$$\begin{aligned} \int_{\mathbb{C}^+} \int_{\mathbb{R}} 1_{|\gamma| < \frac{1}{t}} 1_{|v-\zeta| \leq \frac{1}{t}} \frac{\varrho(v) dv p(d\gamma)}{|v-\zeta-\gamma|^2} &\leq \sup_{|v-\zeta| \leq \frac{1}{t}} \varrho(v) \pi \int_{\mathbb{C}^+} 1_{\text{Im } \gamma < \frac{1}{t}} \frac{p(d\gamma)}{\text{Im } \gamma} \\ &\leq \sup_{|v-\zeta| \leq 1} \varrho(v) \frac{\pi}{t^\epsilon} \int_{\mathbb{C}^+} \frac{p(d\gamma)}{(\text{Im } \gamma)^{1+\epsilon}}. \end{aligned} \quad (\text{II.27})$$

Conversely, the right side is bounded from below for every $T > 0$ by

$$\int_{\mathbb{C}^+} \int_{\mathbb{R}} 1_{|\gamma| < T} 1_{|v-\zeta| \leq 1} \frac{\varrho(v) dv p(d\gamma)}{|v-\zeta-\gamma|^2} \geq c^{-1} \sup_{|v-\zeta| \leq 1} \varrho(v) (1+T)^{-2} \int_{\mathbb{C}^+} 1_{|\gamma| < T} p(d\gamma). \quad (\text{II.28})$$

This yields the result. \square

C. Inverse moments of the Green function for energies within the ac spectrum

In order to apply Lemma II.2 to the proof of our main result one needs to establish the finiteness of the Green function's inverse moments for energies within the ac spectrum, as expressed in (II.17). The starting point for that is the relation:

$$\text{Im } G(0, 0; \zeta) \geq |G(0, 0; \zeta)|^2 \sum_{v \in \mathcal{N}_0} \text{Im } G^{T_0}(v, v; \zeta), \quad (\text{II.29})$$

which follows from (II.19). Among its implications is the zero-one law which was noted and applied in Ref. 4: for each energy $E \in S$ (the full measure subset of \mathbb{R} over which $G(0, 0; E + i0)$ is defined), $\mathbb{P}(\text{Im } G(0, 0; E + i0) \neq 0)$ is either 0 or 1. Equivalently, for any $E \in \sigma_{ac}(H)$, as defined by (I.13): $\mathbb{P}(\text{Im } G(0, 0; E + i0) \leq x) \rightarrow 0$, as $x \rightarrow 0$. The following may be viewed as a quantitative improvement (though not yet the best possible) on that statement.

Theorem II.4. *Under the assumptions of Theorem I.1, for any bounded and measurable set $I \subset \sigma_{ac}(H)$ the function*

$$F(x) := \text{ess sup}_{\zeta \in I+i(0,1]} \mathbb{P}(\text{Im } G(0, 0; \zeta) \leq x) \quad (\text{II.30})$$

satisfies:

$$F(x) \leq Cx^{1+\epsilon} \quad (\text{II.31})$$

for all $x \in [0, x_0]$, at some (finite) $C, x_0, \epsilon > 0$.

Before proving it, let us note that Theorem II.4 ensures the validity of (II.17), since (II.31) implies that for all $\delta \in (0, \epsilon)$:

$$\text{ess sup}_{\zeta \in I+i(0,1]} \mathbb{E}[(\text{Im } G(0, 0; \zeta))^{-1-\delta}] \leq (1+\delta) \int_0^\infty \frac{F(x)}{x^{2+\delta}} dx < \infty. \quad (\text{II.32})$$

We did not push the limits here. In fact, in case $\text{supp } \varrho$ is compact, the subsequent proof shows that $F(x) \leq Cx^\gamma$ for any $\gamma > 0$.

The proof utilizes the observation that by (II.29) the small probability event $\{\text{Im } G(0, 0; E + i0) \leq x\}$ requires the occurrence of K similar and somewhat uncorrelated events $\{|G(0, 0; \zeta)|^2 \text{Im } G^{T_0}(v, v; \zeta) \leq x\}$. Had such a relation been valid with $|G(0, 0; \zeta)|^2$ replaced by a positive constant, it would easily follow that the probability $F(x)$ vanishes faster than any power of x . However, as it is, $|G(0, 0; \zeta)|^2$ can be arbitrarily small. Taking that into account, we get the following relation.

Lemma II.5. *The function $F(x)$ defined in Theorem II.4 is monotone increasing in x , satisfies $\lim_{x \downarrow 0} F(x) = 0$, and*

$$F(x) \leq F(xy^{-2})^K + C(yF(cy))^K + y^r, \quad (\text{II.33})$$

for all $x > 0$ and $y \in (0, y_0)$, with $r = 6$ and some constants $c, C, y_0 \in (0, \infty)$.

Proof. The first statement is implied by monotonicity and the above mentioned 0-1 law (Lemma 4.1 of Ref. 4). To arrive at (II.33) we note that for any $y > 0$ the event in (II.30) occurs only if either $|G(0, 0; \zeta)| \leq y$ or for all $v \in \mathcal{N}_0$: $\text{Im } G^{\mathcal{T}_0}(v, v; \zeta) \leq xy^{-2}$. Introducing the distribution functions $H_\zeta(y) := \mathbb{P}(|G(0, 0; \zeta)| \leq y)$ and $F_\zeta(x) := \mathbb{P}(\text{Im } G(0, 0; \zeta) \leq x)$, we conclude that for all $y > 0$:

$$F_\zeta(x) \leq F_\zeta(xy^{-2})^K + H_\zeta(y). \quad (\text{II.34})$$

To ‘close’ this relation we employ the bound on $H_\zeta(y)$ which is presented in the Lemma II.6, below. Using it, the relation (II.34) yields for F_ζ (and hence also for F) the following non-linear inequality:

$$F_\zeta(x) \leq F_\zeta(xy^{-2})^K + 4K^2 \|\varrho\|_\infty y F_\zeta(2Ky)^K + \mathbb{P}(|V(0)| \geq (4y)^{-1}) \quad (\text{II.35})$$

for any $y \in (0, (4|\zeta|)^{-1})$. Applying to that the Chebychev bound $\mathbb{P}(|V(0)| \geq y^{-1}) \leq y^r \mathbb{E}[|V(0)|^r]$, and the finiteness of $\sup_{\zeta \in I+i(0,1]} |\zeta|$, one may deduce (II.33). \square

In the above proof we relied on the following auxiliary statement.

Lemma II.6. *Under the assumptions of Theorem I.1, for all $x \in (0, \infty)$ one has:*

1. $H_\zeta(x) \leq \mathbb{P}(|V(0)| \geq (4x)^{-1}) + K \mathbb{P}(|G(0, 0; \zeta)| \geq (2Kx)^{-1})$ provided $|\zeta| \leq (4x)^{-1}$.
2. $1 - H_\zeta(x^{-1}) \leq 2\|\varrho\|_\infty x F_\zeta(x)^K$.

Proof. We use the representation (II.19) to conclude:

$$H_\zeta(x) \leq \mathbb{P}\left(|V(0)| + |\zeta| + \sum_{v \in \mathcal{N}_0} |G^{\mathcal{T}_0}(v, v; \zeta)| \geq x^{-1}\right) \leq \mathbb{P}(|V(0)| \geq (4x)^{-1}) + K \mathbb{P}(|G(0, 0; \zeta)| \geq (2Kx)^{-1}). \quad (\text{II.36})$$

Here the second inequality relied on $|\zeta| \leq (4x)^{-1}$ and the fact that $G^{\mathcal{T}_0}(v, v; \zeta)$ with $v \in \mathcal{N}_0$ is identically distributed as $G(0, 0; \zeta)$. Employing (II.19) again we get:

$$\begin{aligned} 1 - H_\zeta(x^{-1}) &= \mathbb{P}(|G(0, 0; \zeta)| > x^{-1}) \leq \mathbb{P}\left(|V(0) - \text{Re } \zeta - \sum_{v \in \mathcal{N}_0} \text{Re } G^{\mathcal{T}_0}(v, v; \zeta)| \leq x \text{ and } \sum_{v \in \mathcal{N}_0} \text{Im } G^{\mathcal{T}_0}(v, v; \zeta) < x\right) \\ &\leq 2\|\varrho\|_\infty x \mathbb{P}\left(\sum_{v \in \mathcal{N}_0} \text{Im } G^{\mathcal{T}_0}(v, v; \zeta) \leq x\right) \leq 2\|\varrho\|_\infty x F_\zeta(x)^K, \end{aligned} \quad (\text{II.37})$$

where the last step is due to the independence of the variables $G^{\mathcal{T}_0}(v, v; \zeta)$, $v \in \mathcal{N}_0$. \square

We now turn to the main result of this section.

Proof of Theorem II.4. The proof proceeds in three steps. We first establish that a power-law bound of the form (II.31) holds with at least a small power. Next, the power is improved to a value greater than 1. That step is simpler for K larger than 2. The remaining case $k = 2$ requires an extra argument, which forms the third step in the proof.

For an initial power bound, we pick $y = x^{1/4}$ in (II.33) to conclude that for all $x \in (0, \infty)$

$$F(x^2) \leq 3 \max\{F(x)^K, C\sqrt{x}\}. \quad (\text{II.38})$$

The convergence $\lim_{x \downarrow 0} F(x) = 0$ implies that there is some $x_0 \in (0, \min\{(9c)^{-4}, 1/2\}]$ such that $F(x) \leq 1/2$ for all $x \in [0, x_0]$. Hence there is some $\alpha_0 \in (0, 1/8]$ such that

$$F(x_0) \leq \frac{x_0^{2\alpha_0}}{3}. \quad (\text{II.39})$$

We now define recursively $x_n := x_{n-1}^2$ for all $n \in \mathbb{N}$. By induction on n , one establishes that $F(x_n) \leq \frac{1}{3}x_{n+1}^{\alpha_0}$ for all $n \in \mathbb{N}_0$. Namely, for $n = 0$ this is the content of (II.39). For the induction step, we use (II.38) which implies

$$\frac{F(x_n)}{x_{n+1}^{\alpha_0}} \leq 3 \max\left\{\frac{F(x_{n-1})^K}{x_n^{2\alpha_0}}, Cx_n^{\frac{1}{2}-2\alpha_0}\right\} \leq 3 \max\left\{\left(\frac{F(x_{n-1})}{x_n^{\alpha_0}}\right)^2, cx_0^{\frac{1}{4}}\right\} \leq \frac{1}{3}. \quad (\text{II.40})$$

Since F is monotone increasing, this implies that for all $x \in (x_{n+1}, x_n]$ and all $n \in \mathbb{N}$, and hence for all $x \in (0, x_0]$:

$$F(x) \leq F(x_n) \leq \frac{x_{n+1}^{\alpha_0}}{3} \leq \frac{x^{\alpha_0}}{3}. \quad (\text{II.41})$$

This completes the proof of the initial (still insufficient) bound.

In the second step in the proof, we will improve on the power law with which (II.39) holds. To this end, suppose that for some $\alpha, C > 0$ and all $x \in (0, x_0]$:

$$F(x) \leq C x^\alpha \quad (\text{II.42})$$

Then (II.33) with $y = x^{\frac{\alpha K}{1+\alpha K}}$ implies that for all $\alpha \in (0, 5/4]$ and $K \geq 2$:

$$F(x) \leq C \left(x^{\alpha K \frac{1+\alpha K}{1+3\alpha K}} + x^{\frac{r\alpha K}{1+\alpha K}} \right) \leq C \left(x^{\alpha \frac{2K}{5}} + x^{\alpha \frac{7r}{4}} \right). \quad (\text{II.43})$$

with some constant $C > 0$. In case $K \geq 3$, this shows that one may improve the exponent α in the bound (II.42) at least by a factor of two as long as $\alpha \leq 5/4$. This proves that the bound (II.42) holds with $\alpha = 5/4$ if $K \geq 3$.

In case $K = 2$, we need to improve on the non-linear inequality (II.33) in order to improve on the apriori bound. To do so we will denote by $x_{\pm 1}$ the two neighbors of the root and expand the recursion relation (II.29) one step further:

$$\begin{aligned} \text{Im } G(0, 0; \zeta) &\geq \sum_{\nu \in \{\pm 1\}} |G(0, x_\nu; \zeta)|^2 \sum_{v \in \mathcal{N}_{x_\nu} \setminus \{0\}} \text{Im } G^{\mathcal{T}_{x_\nu}}(v, v; \zeta) \\ &\geq \sum_{\nu \in \{\pm 1\}} \frac{|\Gamma(x_\nu)|^2}{(|V(0)| + |\zeta| + |\Gamma(x_\nu)| + |\Gamma(x_{-\nu})|)^2} \sum_{v \in \mathcal{N}_{x_\nu} \setminus \{0\}} \text{Im } G^{\mathcal{T}_{x_\nu}}(v, v; \zeta). \end{aligned} \quad (\text{II.44})$$

where $\Gamma(x_\nu) := G^{\mathcal{T}_0}(x_\nu, x_\nu; \zeta)$. We pick $y > 0$ and condition on the event $E_0 := \{|V(0)| \leq y^{-1}\}$. We now distinguish three events, which decompose the probability space:

1. $E_1 := \{|\Gamma(x_\nu)| > y^{-1} \text{ for both } \nu = \pm 1\}$.
2. $E_2 := \{|\Gamma(x_1)| > y^{-1} \text{ and } |\Gamma(x_{-1})| \leq y^{-1}\}$, or vice versa in case E_3 .
3. $E_4 := \{|\Gamma(x_\nu)| \leq y^{-1} \text{ for both } \nu = \pm 1\}$.

The probability of the event E_1 is estimated with the help of Lemma II.6 below:

$$\mathbb{P}(E_1) = \mathbb{P}(|\Gamma(x_1)| > y^{-1})^2 \leq C y^2 F_\zeta(y)^4. \quad (\text{II.45})$$

In the event E_2 one of the quadratic factors in the first sum in (II.44) is bounded from below,

$$\frac{|\Gamma(x_1)|}{(|V(0)| + |\zeta| + |\Gamma(x_1)| + |\Gamma(x_{-1})|)} \geq \frac{y^{-1}}{y^{-1} + |\zeta| + 2y^{-1}} = \frac{1}{3 + y|\zeta|}. \quad (\text{II.46})$$

Supposing that y is upper bounded, there is some constant $C > 0$ such that we may hence estimate

$$\mathbb{P}(\{\text{Im } G(0, 0; \zeta) \leq x\} \cap E_0 \cap E_2) \leq \mathbb{P}\left(\sum_{v \in \mathcal{N}_{x_1} \setminus \{0\}} \text{Im } G^{\mathcal{T}_{x_1}}(v, v; \zeta) \leq Cx\right) \leq F_\zeta(Cx)^2. \quad (\text{II.47})$$

In case $E_0 \cap E_3$ happens, one proceeds analogously. It therefore remains to estimate the last case in which we use (II.29) and the fact that $|G(0, 0; \zeta)| \leq (y^{-1} + |\zeta| + 2y^{-1})$ in the event $E_0 \cap E_4$. There is hence some constant $C > 0$ such that

$$\mathbb{P}(\{\text{Im } G(0, 0; \zeta) \leq x\} \cap E_0 \cap E_4) \leq \mathbb{P}\left(\sum_{v \in \mathcal{N}_0 \setminus \{0\}} \text{Im } G^{\mathcal{T}_0}(v, v; \zeta) \leq Cxy^{-2}\right) \leq F_\zeta(Cxy^{-2})^2. \quad (\text{II.48})$$

Summarizing, we also have for all sufficiently small $y > 0$:

$$F(x) \leq F(Cxy^{-2})^2 + C(y^2 F(y)^4 + y^r + 2F(Cx)^2). \quad (\text{II.49})$$

We now proceed as in the case $K \geq 3$ and assume a bound of the form (II.42). Picking $y = x^{\frac{\alpha}{1+4\alpha}}$ then yields

$$F(x) \leq C \left(x^{\alpha \frac{2+4\alpha}{1+4\alpha}} + x^{\alpha \frac{r}{1+4\alpha}} + x^{2\alpha} \right). \quad (\text{II.50})$$

Since $r = 6$ this yields an improvement of the exponent in (II.42) provided $\alpha < 5/4$. This completes the proof of the last assertion in Theorem II.4 also for $K = 2$. \square

Appendix A: On the two notions of ac spectrum

As mentioned in the introduction, there is more than one natural choice for the notion of the absolutely continuous spectrum of an ergodic self adjoint operator H , and in particular of the spectral measure associated with the vector δ_0 . In addition to $\sigma_{ac}(H)$ of (I.13), one may also consider $\Sigma_{ac}(H)$ which is defined as the minimal closed set in which the ac component of the spectral measure is supported (or equivalently: the collection of energies for which the ac spectral measure of arbitrarily small neighborhoods is non zero). For better clarity in regard to the statements proven above, let us add some general comments on the relation between the two sets $\sigma_{ac}(H)$ and $\Sigma_{ac}(H)$.

1. It may well be that random Schrödinger operators have only ac and pp spectra, with $\sigma_{ac}(H)$ a finite union of open intervals and $\Sigma(H)$ being the disjoint union of Σ_{pp} and σ_{ac} . In that case, the only difference between $\sigma_{ac}(H)$ and $\Sigma_{ac}(H)$ is the inclusion of the endpoints of the intervals. However, the techniques available so far, including the criteria which in terms of the function $\varphi(E; 1) < \log K$, whose continuity and other regularity properties are still not sufficiently understood, do not yet allow such a conclusion.
2. The measurable set $\sigma_{ac}(H)$ is dense within the closed set $\Sigma_{ac}(H)$.
3. Despite the above point it is not generally true, within the context of self adjoint operators, that the set difference between $\Sigma_{ac}(H)$ and $\sigma_{ac}(H)$ is of measure zero. (For a counterexample consider a self adjoint operator for which $\sigma_{ac}(H)$ is a union of a countable collection of intervals in $[0, 1]$, the sum of whose measures is smaller than 1, yet whose centers form a dense subset of $[0, 1]$.) Hence, even though for every $E \in \sigma_{ac}(H)$:

$$\mathbb{P}(\text{Im } G(0, 0; E + i0) \neq 0) > 0. \quad (\text{A.1})$$

one *cannot* conclude that (A.1) holds also for almost every $E \in \Sigma_{ac}(H)$.

4. Another relevant property which is not automatically shared by $\Sigma_{ac}(H)$ is that within $\sigma_{ac}(H)$ the spectrum is (almost surely) *purely* ac (the mean value of the singular measure of $\sigma_{ac}(H)$ is zero, though that is not known for $\Sigma_{ac}(H)$). This follows by the arguments found in Refs. 6 and 24, combined with the 0-1 law which is valid for tree graphs by which (A.1) implies the stronger statement that the probability seen there is actually 1.

The point 3. is the main reason our results are formulated for $\sigma_{ac}(H)$ rather than $\Sigma_{ac}(H)$.

Appendix B: Proof of a (known) ballistic upper bound

For completeness of the presentation, following is a simple proof of a known ballistic upper bound which our main result complements.

Proof of (I.6). For an arbitrary graph \mathcal{G} , on which the distance is denoted by $d(x, y)$, let A be an operator on $\ell^2(\mathcal{G})$ with kernel $A(x, y)$, for which

$$g(\alpha) := \sup_{x \in \mathcal{G}} \sum_{y \in \mathcal{G}} |A(x, y)| e^{\alpha d(x, y)} < \infty \quad (\text{B.1})$$

for all $\alpha < \infty$ (which implies that g is continuous). For such operators the exponential function admits a convergent power series expansion. Using it one finds, for each $x_0 \in \mathcal{G}$:

$$\sum_{x \in \mathcal{G}} |\langle \delta_x, e^{-itA} \delta_{x_0} \rangle| e^{\alpha d(x, y)} \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{j=1}^n \sum_{x_j \in \mathcal{G}} [|A(x_{j-1}, x_j)| e^{\alpha d(x_{j-1}, x_j)}] \leq e^{tg(\alpha)}, \quad (\text{B.2})$$

where use was made of the triangle inequality: $d(x, x_0) \leq \sum_{j=1}^n d(x_j, x_{j-1})$.

Next, it may be noted that the above bound extends unchanged to operators of the form $H = A + V$, with A as above and V an arbitrary real valued potential, that is a self adjoint operator with a diagonal kernel. Applying the bound (B.2) within the Lie-Trotter formula

$$\langle \delta_x, e^{-itH} \delta_{x_0} \rangle = \lim_{k \rightarrow \infty} \langle \delta_x, [e^{-iAt/k} e^{-iVt/k}]^k \delta_{x_0} \rangle \quad (\text{B.3})$$

we get:

$$\sum_{x \in \mathcal{G}} |\langle \delta_x, e^{-itH} \delta_{x_0} \rangle| e^{\alpha d(x,y)} \leq \lim_{k \rightarrow \infty} \left[e^{g(\alpha)t/k} \right]^k = e^{t g(\alpha)}. \quad (\text{B.4})$$

Let now $\hat{v} := \min_{\alpha > 0} g(\alpha)/\alpha$ and denote by $\mu \in (0, \infty)$ the (largest) minimizing α (both are easily seen to be well defined). Applying the Chebyshev inequality to the probability of the event and using the fact that $|\langle \delta_x, e^{itH} \delta_{x_0} \rangle| \leq 1$ one gets, for any $v > \hat{v}$:

$$\begin{aligned} \Pr_{\delta_0, t}(|x| > vt) &\leq e^{-\mu tv} \sum_{x \in \mathcal{G}} \Pr_{\delta_0, t}(x) e^{\mu|x|} \leq e^{-\mu tv} \sum_{x \in \mathcal{G}} |\langle \delta_x, e^{-itH} \delta_0 \rangle| e^{\mu|x|} \\ &\leq e^{-\mu tv} e^{t g(\mu)} = e^{-\mu t[v - \hat{v}]}, \end{aligned} \quad (\text{B.5})$$

where $|x| := d(x, 0)$. This directly implies the ballistic upper bound (I.6) for the operator H of (I.1). □

Let us note that a generalization of the upper bound (I.6) (though not of the lower bound which is our main result) can be found in the Lieb-Robinson theorem on the finiteness of the speed of sound in many body quantum systems²⁰, a general result which is of current interest.²¹ In fact, Eq. (I.6) can also be concluded from the latter through the second quantization formalism in the context of a system of non-interacting Fermions.

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